Solutions to Homework 5 - Markov Chains

1) Stationary distribution. Consider a Markov chain (MC) $X_N = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/4 & 3/4 \\ 1/5 & 4/5 \end{pmatrix}.$$

To obtain the stationary distribution $\pi = [\pi_1; \pi_2$

$$\lim_{n\to\infty} P^n_{22} = \pi_2 = \frac{15}{19}.$$

Finally, by virtue of the ergodic theorem the time average $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} I\{X_k = 1\}$ also converges. The long-run fraction of time the MC visits state 1 is thus given by

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} I\{X_k = 1\} = \pi_1 = \frac{4}{19}.$$

2) A cloudy town. A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities.

Let $X_N = X_0, X_1, \ldots, X_n, \ldots$ be the random process describing the weather evolution of the given town, with n denoting the day number. Given the nature of the evolution of the process (meaning the weather today only

Customers who do not nd the product in stock depart without making a purchase. The store orders $q > 0$ new units of the product from its supplier at the end of the day (after that day's demand has materialized). However, the supplier is not completely reliable, and each day, with probability independent of everything else, the order is permanently lost in which case the order does not arrive to the store. If the order is not lost, it arrives to the

Now, what is left is to carefully put all pieces (2)-(6) together back in (1). For all $i \ge 0$, the transition probabilities are thus given by

$$P_{i0} = \sum_{k=i}^{\infty} p(k);$$

$$P_{ij} = \begin{cases} p(i-j); & i \ge j \\ 0; & i < j \end{cases}; \quad \text{for } 0 < j < q;$$

$$P_{iq} = \begin{cases} p(i-q) + (1\ \ )\sum_{k=i}^{\infty} p(k); & i \ge q \\ (1\ \ )\sum_{k=i}^{\infty} p(k); & i < q \end{cases};$$

$$P_{ij} = \begin{cases} p(i-j) + (1\ \ )p(i-j+q); & i-j \ge 0 \\ (1\ \ )p(i-j+q); & i-j+q \ge 0;\ i-j < 0 \\ 0; & i-j+q < 0 \end{cases}; \quad \text{for } j > q.$$

4) Non-invertible function of a Markov chain. Suppose that $X_N = \{X_0; X_1; \ldots; X_n; \ldots\}$ is a MC with state space $S = \{1;2;3\}$, transition probability matrix

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \\ 3/4 & 1/4 & 0 \end{pmatrix}$$

and initial distribution $P[X_0 = 1] = 1/5$, $P[X_0 = 2] = 2/5$ and $P[X_0 = 3] = 2/5$. Suppose that the random process $Y_N = \{Y_0; Y_1; \ldots; Y_n; \ldots\}$ satisfies $Y_n = g(X_n)$, $n \ge 0$, where $g(1) = 1$ and $g(2) = g(3) = 2$.

To calculate $P(Y_2 = 1 \mid Y_1 = 2; Y_0 = 1)$, one can resort to the definition of conditional probability and write

$$\begin{aligned} P(Y_2 = 1 \mid Y_1 = 2; Y_0 = 1) &= \frac{P[Y_2 = 1; Y_1 = 2; Y_0 = 1]}{P[Y_1 = 2; Y_0 = 1]} \\ &= \frac{P[X_2 = 1; X_1 \in \{2;3\}; X_0 = 1]}{P[X_1 \in \{2;3\}; X_0 = 1]} \\ &= \frac{P[X_2 = 1; X_1 = 2; X_0 = 1] + P[X_2 = 1; X_1 = 3; X_0 = 1]}{P[X_1 = 2; X_0 = 1] + P[X_1 = 3; X_0 = 1]} \\ &= \frac{P[X_0 = 1]\,P_{12}P_{21} + P[X_0 = 1]\,P_{13}P_{31}}{P[X_0 = 1]\,P_{12} + P[X_0 = 1]\,P_{13}} = \frac{5}{12}. \end{aligned}$$

Because the function g is not invertible, it turns out that Y_N is not Markov chain. Actually, one can verify that

$$P(Y_2 = 1 \mid Y_1 = 2; Y_0 = 1) \ne P(Y_2 = 1 \mid Y_1 = 2) = \frac{17}{36}.$$

5) A non-irreducible Markov chain. Consider a MC $X_N = \{X_0; X_1; \ldots; X_n; \ldots\}$ with state space $S = \{1;2;3\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/3 & 1/6 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A) The

Fig. 1. State transition diagram for the MC with transition probability matrix $\mathbf{P}$. The MC has three communication classes.

Because state 1 is transient, one has $P_{11}^{\infty} = 0$. To compute P_{12}^{∞} and P_{13}^{∞}, introduce the matrix of limiting probabilities

$$\mathbf{P}^{\infty} = \begin{pmatrix} 0 & P_{12}^{\infty} & P_{13}^{\infty} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that must be a fixed point of the recursion $\mathbf{P}^{n} = \mathbf{P}\,\mathbf{P}^{n-1}$, implying $\mathbf{P}^{\infty} = \mathbf{P}\,\mathbf{P}^{\infty}$. This identity gives the equations required to determine P_{12}^{∞} and P_{13}^{∞}, which yield

$$\frac{P_{12}^{\infty}}{3} + \frac{1}{6} = P_{12}^{\infty} \;\Rightarrow\; P_{12}^{\infty} = \frac{1}{4}$$

$$\frac{P_{13}^{\infty}}{3} + \frac{1}{2} = P_{13}^{\infty} \;\Rightarrow\; P_{13}^{\infty} = \frac{3}{4}.$$

All in all, the matrix of limiting probabilities is

$$\mathbf{P}^{\infty} = \begin{pmatrix} 0 & 1/4 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

C) The matrix of limiting probabilities $\mathbf{P}^{\infty}$ suggests the following three stationary distributions

$$\pi_1 = [0, 1/4, 3/4]^T, \quad \pi_2 = [0, 1, 0]^T, \quad \pi_3 = [0, 0, 1]^T.$$

It is straightforward to check that $\pi_i = \mathbf{P}^T \pi_i$, for each $i = 1, \ldots, 3$.

D) From the first row of $\mathbf{P}^{\infty}$, one can claim that given $X_0 = 1$ the MC will end up in state 2 (and stay there forever) with probability $1/4$, or else end up in state 3 (and stay there forever) with probability $3/4$. This observation immediately leads to the conclusion that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{I}\{X_k = 2\}$$

will almost surely converge to a random variable that is Bernoulli distributed with parameter $1/4$.

6) A null-recurrent Markov chain. Consider a MC with state space $\mathcal{S} = \{1, 2, \ldots\}$ and transition probabilities $P_{i,i+1} = i/(i+1)$ and $P_{i1} = 1/(i+1)$ for $i = 1, 2, \ldots$. The transition probability matrix is

$$\mathbf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/3 & 0 & 2/3 & 0 & 0 & \cdots \\ 1/4 & 0 & 0 & 3/4 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

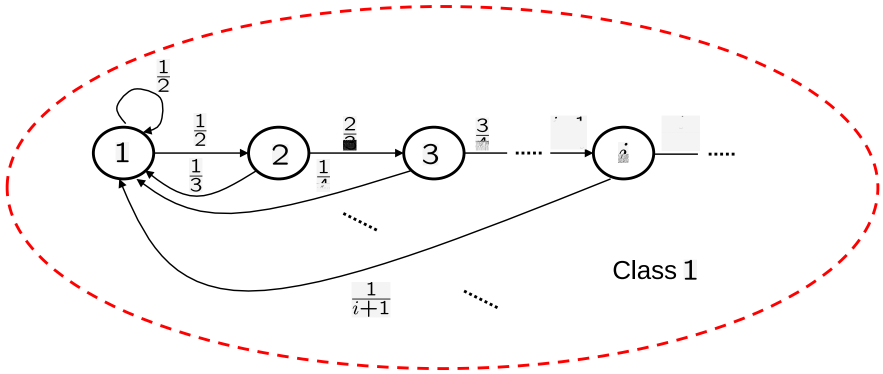


Fig. 2. State transition diagram for the MC with transition probability matrix $\mathbf{P}$. The MC has three communication classes.

The state transition diagram is depicted in Fig. 2, from where it is apparent that state 1 is accessible from all other states (via single-step transitions). Moreover, all other states are clearly accessible from state 1. This shows that all states communicate and form a single (infinitely large) class, so the MC is irreducible.

The goal is to establish the recurrence properties of the MC. Because the MC is irreducible and recurrence is a class property, it suffices to analyze the recurrence properties of a single state, say state 1. To that end, define the return time T_1 to state 1 as

$$T_1 = \min\{n > 0 \mid X_n = 1\}$$

State 1 (and hence the MC) will be recurrent if $P\left(T_1 < \infty \mid X_0 = 1\right) = 1$, which is of course equivalent to $P\left(T_1 = \infty \mid X_0 = 1\right) = 0$. As it can be readily seen from the state transition diagram, the probability that the MC never returns to state 1 given that it started in that state is given by

$$\begin{aligned} P\left(T_1 = \infty \mid X_0 = 1\right) &= P_{12} \times P_{23} \times P_{34} \times \cdots \times P_{i,i+1} \times \cdots \\ &= \lim_{n\to\infty} \prod_{i=1}^{n} P_{i,i+1} = \lim_{n\to\infty} \prod_{i=1}^{n} \frac{i}{i+1} = \lim_{n\to\infty} \frac{1}{n+1} = 0. \end{aligned}$$

In obtaining the second last inequality we have used the fact that terms in successive products of probabilities cancel out, and only the numerator from the first and denominator from the last probability survive. This establishes that the MC is recurrent as desired.

To further show that it is null recurrent, it suffices to focus on first state and verify that $E\left[T_1 \mid X_0 = 1\right] = \infty$. Recalling the definition of T_1, one can obtain the relevant conditional pmf

$$P[T_1 = n \mid X_0 = 1] = \begin{cases} P_{11}, & n = 1 \\ P_{12} \times P_{21}, & n = 2 \\ P_{12} \times P_{23} \times P_{31}, & n = 3 \\ \vdots & \vdots \\ P_{12} \times P_{23} \times \cdots \times P_{i-1,i} \times P_{i1}, & n = i \\ \vdots & \vdots \end{cases}$$

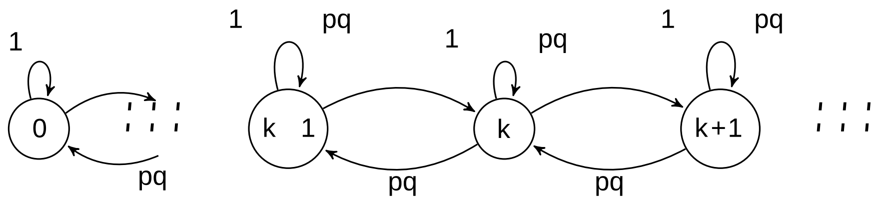
whose general term can be simplified as

$$P\left(T_1 = n \mid X_0 = 1\right) = \left(\prod_{i=1}^{n-1} P_{i,i+1}\right) P_{n1} = \left(\prod_{i=1}^{n-1} \frac{i}{i+1}\right) \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

The conditional expectation is from the definition

$$E\left[T_1 \mid X_0 = 1\right] = \sum_{n=1}^{\infty} n\, P\left(T_1 = n \mid X_0 = 1\right) = \sum_{n=1}^{\infty} n \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

The infinite sum diverges, establishing that state 1 (hence, the MC) is null-recurrent as desired.



1
0
pq
1
pq
k 1
1
pq
k
pq
1
pq
k+1
pq

TABLE I

SUMMARY OF THE RECURRENCE ANALYSIS RECURRENCE DEPENDS ON THE PARAMETERS AND pq.

Parameters	Effect on recurrence
$= 0, pq = 0$	Every state is a class. Every class is positive recurrent.
$= 0, pq > 0$	Every state is a class. Class 0 is positive recurrent. All other classes are transient.
$> 0, pq = 0$	Every state is a class. Every class is transient.
$> pq > 0$	The MC is irreducible and transient.
$pq > \quad > 0$	ys

In order to obtain the value of the constant c, we use the fact that the limit probabilities $_k$ of every state $k = 0; 1; 2; \ldots$ in the MC must add up to 1. Formally,

$$1 = \sum_{k=0}^{\infty} {}_k = \sum_{k=0}^{\infty} c \left(\frac{}{pq}\right)^k = c \sum_{k=0}^{\infty} \left(\frac{}{pq}\right)^k = c \frac{1}{1 \quad \frac{}{pq}}; \qquad (20)$$

where we used the fact that $< pq$ for the convergence of the infinite geometric series. From (20) we obtain $c = 1 \quad =pq$ resulting in the limit probabilities

$$_k = c^{\;k} = \left(1 \quad \frac{}{pq}\right)\left(\frac{}{pq}\right)^k : \qquad (21)$$

Observe that in (21), as tends to pq from below, probabilities $_k$ tend to 0 for all k, which is consistent with the fact that when $= pq$ the MC is null recurrent. Similarly, when $> pq > 0$ and the MC is transient, probabilities $_k = 0$

which is true for $|x| < 1$ and that we used in (20). After differentiating both sides of (28) and multiplying both of them by x, we obtain

$$\sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}. \tag{29}$$

By using (29) for $x = \lambda/pq$, we can evaluate the expected value expression in (27), to obtain

$$\lim_{n\to\infty} E[Q_{jn}] = \left(1-\frac{\lambda}{pq}\right)\frac{\lambda/pq}{\left(1-\frac{\lambda}{pq}\right)^2} = \frac{\lambda/pq}{1-\frac{\lambda}{pq}} = \frac{\lambda}{(pq-\lambda)}. \tag{30}$$

Notice that the expression obtained in (30) for the expected queue length is quite intuitive. The expected length grows with increasing arrival rate λ and decreases when the difference between the successful transmission rate pq and the arrival rate λ increases.

E) Probability of successful transmission and optimal transmission probability p. So far, we have assumed that the probability q that a transmission by an arbitrary terminal j does not experience a collision with any other terminal is given. However, under the dominant system assumption (B), for a transmitted packet not to collide it must be that none of the remaining $J-1$ transmitted any packet. Since every terminal acts independently, we may write

$$q = (1-p)^{J-1}. \tag{31}$$

This allows us to compute the probability p that maximizes the probability of successful transmission pq. In order to do this, we differentiate pq with respect to p and look for the roots of the corresponding equation, i.e.

$$\frac{d}{dp}[pq] = \frac{d}{dp}\left[p(1-p)^{J-1}\right] = (1-p)^{J-1} - (J-1)p(1-p)^{J-2} = 0. \tag{32}$$

Since we obviously assume $p < 1$ because otherwise the probability of successful transmission is trivially null, we may divide (32) by $(1-p)^{J-2}$ to obtain a linear expression that yields the optimal probability

$$p^* = \frac{1}{J}. \tag{33}$$

Since the probability of the queue being empty is an increasing function of pq [cf. (22)] and the expected queue length is a decreasing function of pq [cf. (30)], the optimal probability p^* in (33) entails simultaneously shorter queues and higher probabilities of these queues being empty. Moreover, for p^*, the probability of no collision becomes [cf. (31)]

$$q^* = \left(1-\frac{1}{J}\right)^{J-1}. \tag{34}$$

If we consider a system with many terminals, we may estimate the probability of no collision as the limit of q^* when J tends to infinity, i.e.,

$$\lim_{J\to\infty} q^* = \lim_{J\to\infty}\left(1-\frac{1}{J}\right)^{J-1} = \lim_{J\to\infty}\left(1-\frac{1}{J}\right)^{J}\left(1-\frac{1}{J}\right)^{-1} = \frac{1}{e} \approx 0.368. \tag{35}$$

The above result implies that RA communications utilizes approximately 37% of the available access point resources without any coordination overhead among terminals.

F) Average time occupancies. It is possible to argue that the limit probabilities in (21) as well as the performance indicators in (22)-(24), and (27) are of little practical value. What these probabilities express is an average across all possible paths of the communication system. Say we run the system once and obtain a certain path $Q_{jn}^{(1)}$, a

Fig. 4. Evolution of the first four queues over the first 1000 time slots. Queue 3 achieves a maximum queue length of seven packets. Every other queue length remained under this value.

these probabilities cannot be computed in closed form. However, we may use our simulation results to estimate the probability distribution function in (38). Moreover, since the MC is ergodic, we may compute the probabilities in (38) as the time averages over a single simulation run, that is

$$ {}_k \quad \frac{1}{N}\sum_{n=1}^{N} I\{R_{jn} = k\}; \quad \text{for all } k; \tag{39} $$

and for large N. The Matlab script to generate the requested plot follows:

```
clear all; close all; clc;

J=16;
p=1/J;
N=10^5;
lambda=0.9*p*(1-p)^(J-1);

x = aloha_uplink_simulation(J,p,lambda,N);

Q1=max(x(1,:));
frequencies=zeros(1,Q1+1);
for i=0:Q1
    frequencies(1,i+1)=sum(x(1,:)==i);
end

rho=lambda/(p*(1-p)^(J-1));

A=[frequencies/N;(1-rho)*(rho.^(0:Q1))];
A=A';
figure
```