ECE440 - Introduction to Random Processes

Midterm Exam

November 7, 2016

Instructions:

This is an open book, open notes exam.
Calculators are not needed; laptops, tablets and cell-phones are not allowed.
Perfect score: 100 (out of 103, extra points are bonus points).
Duration: 75 minutes.
This exam has 12 numbered pages, check now that all pages are present.
Make sure you write your name in the space provided below.
Show all your work, and write your final answers in the boxes when provided.

Name:________________________________

Problem	Max. Points

1. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$P = \begin{pmatrix} 1/5 & 4/5 \\ 2/5 & 3/5 \end{pmatrix}$$

and initial distribution $P(X_0 = 1) = 3/4$ and $P(X_0 = 2) = 1/4$. To spare you of pointless

(d) (10 points) Compute the stationary distribution of X_N.

(e) (4 points) Calculate

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} I\{X_i = 2\}$$

and provide justification for the existence of the limit.

(c) (4 points) $\mathrm{P}\ \ X = 1\ \ Y = 3\ \ = ?$

(d) (5 points) $\mathrm{E}\ \ X\ \ Y = 3\ \ = ?$

3. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$P = \begin{pmatrix} 0.2 & 0 & 0 & 0.8 & 0 \\ 0.3 & 0.5 & 0.2 & 0 & 0 \\ 0.6 & 0.3 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) (8 points) Draw the corresponding state transition diagram.

(b) (3 points) Is state 4 aperiodic? Explain.

(c) (4 points) $\lim_{n \to \infty} P\left(X_n = 3 \mid X_0 = 1\right) = ?$

(d) (5 points) Explain in a few sentences why, in the long run, you would expect to find the Markov chain in state 5 half of the time.

4. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables with mean $\mathbb{E}[X_1] = 5$ and variance $\operatorname{var}[X_1] = 1$. Consider the following random variables

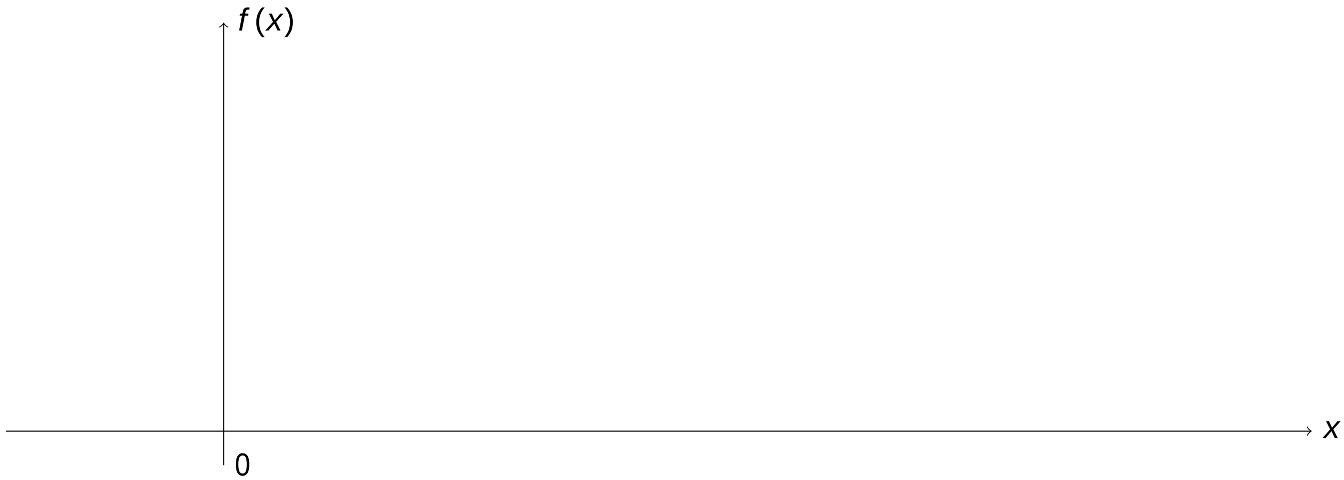
$$S_n := \sum_{i=1}^{n} X_i,$$

$$\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{S_n}{n},$$

$$Z_n := \frac{\sum_{i=1}^{n} X_i - 5n}{\sqrt{n}} = \frac{S_n - 5n}{\sqrt{n}}.$$

Suppose that $n = 100$ is large enough so that limiting behaviors become apparent.

(a) (12 points) Sketch the probability density functions (pdfs) of S_{100}, $\bar{X}_{100}$, and Z_{100}, superimposing the three plots in the set of axes provided below. Only rough, qualitative depictions are required, focusing on the notable values where the pdfs are centered, and their relative widths and heights. Justify your answer and label your plots.



(b) (2 points) Will your plots fundamentally change if the common distribution of the random variables $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ differs from that of (a), while the mean and variance remain the same (that is, one still has $\mathrm{E}[X_1] = 5$ and $\mathrm{var}[X_1] = 1$)? Explain.

5. Old McDonald had a farm, but now he runs a monster-sighting business in Loch Ness, Scotland. Every day, he is unable to run the boat tour due to bad weather with probability p, independently of all other days. McDonald works every day except the bad-weather days, which he takes as holiday.

Let Y be the number of consecutive days McDonald has to work between bad-weather days. Let X be the total number of customers who go on McDondald's boat trip in this period of Y days. Conditioned on Y, the distribution of X is $X \quad Y \quad$ Poisson$(\ Y)$, meaning the conditional probability mass function is given by

$$\mathrm{P}\ \ X = x \ \ Y = y \ \ = \frac{(\ y)^x e^{\ \ y}}{x!}.$$

(a) (6 points) $\mathrm{E}[Y] = ?$ [Hint: Argue that the random variable $Z := Y + 1 \quad$ Geometric(p)]

(b) (4 points) $\mathrm{E} \quad X \quad Y = y \quad = ?$

(c) (5 points) $\mathrm{E}[X] = ?$

6. Suppose that X_n is the amount of inventory in a store at the beginning of the time period n. At the beginning of each period, the inventory decreases by one unit provided the inventory level is positive, and otherwise the inventory remains at 0 until the end of the period. At the end of period n, the inventory is replenished by an amount R_n, where $R_{\mathrm{N}} = R_0; R_1; : : : ; R_n; : : :$ is an i.i.d. sequence (independent of X_0) of non-negative integer-valued random variables, each with probability mass function $q(\ )$; i.e.,

$$\mathrm{P}\,(R_0 = i) = q(i); \quad i \, i \, \mathrm{n}$$

(b) (6 points) Determine the transition probabilities P_{ij} for all i 2 and j i 1.

(c) (6 points) Determine the transition probabilities P_{ij} for all i 2 and 0 $j < i$ 1.