ECE440 - Introduction to Random Processes

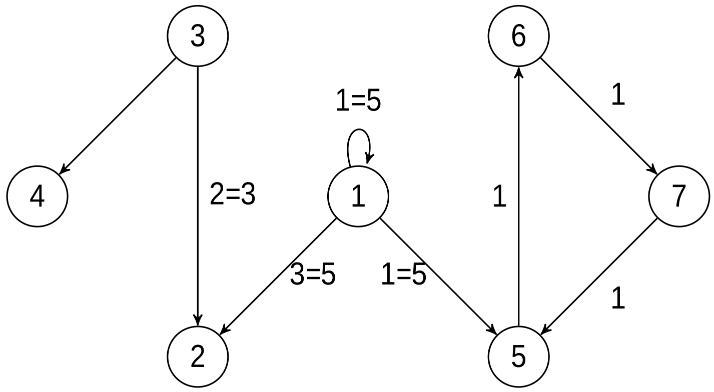
Midterm Exam

November 1, 2021

Instructions:

This is an open book, open notes exam.
Calculators are not needed; laptops, tablets and cell-phones are not allowed.
Perfect score: 100 points.
Duration: 90 minutes.

1. Consider a Markov chain $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$, state transition diagram



(c) (6 points) What is the period of state 6?

(d) (4 points) $\lim_{n \to \infty} P_{11}^{n} = ?$

(e) (6 points) Calculate

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=2}^{n} I \quad X_i = 7 \quad X_1 = 5$$

and provide justification for the existence of the limit.

2. (10 points) Consider i.i.d. continuous random variables $X_1, \ldots, X_{10}$ with probability density function

$$f_X(x) =$$

3. Draw a county at random from the United States. Then draw n people at random from that county. Let $0 \quad X \quad n$ be the number of those people who are infected with COVID-19. If Q denotes the proportion of people in the county with the virus, then Q is also a random variable since it varies from county to county. Given $Q = q$, we have that X

(c) (6 points) $\mathrm{var}[X] = ?$

4. Suppose that $X_{\mathbb{N}} = X_1, X_2, \ldots, X_n, \ldots$ is an i.i.d. sequence of random variables, which are uniformly distributed in the interval $[0, 1]$.

(a) (8 points) Define the random variable

$$Y = \min\{X_1, X_2\}.$$

Write down an expression for $f_Y(y)$, the probability density function of Y. [Hint: it might be easier to first compute $\mathrm{P}(Y > y)$.]

(b) (4 points) Let $Y_{\mathbb{N}} = Y_1, Y_2, \ldots, Y_n, \ldots$ be the sequence of random variables given by

$$Y_n = \min\{X_1, \ldots, X_n\}, \quad n \geq 1.$$

Show that Y_n converges in probability to 0 as $n \to \infty$.

5. Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2\}$, transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix};$$

where $0 < a < 1$ and $0 < b < 1$. We define the recurrence time of state $i \in S$ as

$$T_i = \min\{n > 0 : X_n = i\} \text{ given that } X_0 = i.$$

Accordingly, T_i

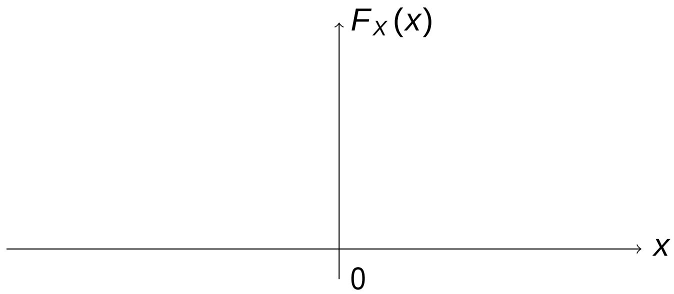
(b) (6 points) $E\left(T_1 \mid X_0 = 1\right) = ?$

6. Suppose that we want to evaluate the integral

$$I = \int_a^b f(x)dx$$

for some integrable function f. Unlike polynomial, rational or trigonometric functions, if f is complicated then there may be no known closed form expression for I

7. (8 points) Consider a random variable X with cumulative distribution function $F_X(x) = P(X \quad x)$ given in the following figure.



8. (12 points) Suppose that $X_{\mathbb{N}} = X_0, X_1, \ldots, X_n, \ldots$ is a Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition probability matrix

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $0 < p < 1$ and $q = 1 - p$.

Determine the stationary distribution of $X_{\mathbb{N}}$. [Reminder: for your calculations, it might useful to recall the partial geometric sum $\sum_{r=0}^{k} \alpha^r = \frac{1-\alpha^{k+1}}{1-\alpha}$, for $\alpha \neq 1$.]