Gaussian, Markov and stationary processes

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Introduction and roadmap

Random processes

- Random processes assign a function $X(t)$ to a random event
    - Without restrictions, there is little to say about them
    - Markov property simplifies matters and is not too restrictive
- Also constrained ourselves to discrete state spaces
    - Further simplification but might be too restrictive
- Time t and range of $X(t)$ values continuous in general
    - Time and/or state may be discrete as particular cases
- Restrict attention to (any type or a combination of types)
    - Markov processes (memoryless)
    - Gaussian processes (Gaussian probability distributions)
    - Stationary processes ("limit distribution")

Markov processes

- $X(t)$ is a Markov process when the future is independent of the past
- For all $t > s$ and arbitrary values $x(t)$, $x(s)$ and $x(u)$ for all $u < s$

$$P\big[X(t) \leq x(t) \,\big|\, X(s) \leq x(s); X(u) \leq x(u); u < s\big]$$
$$= P\big[X(t) \leq x(t) \,\big|\, X(s) \leq x(s)\big]$$

  - ⇒ Markov property defined in terms of cdfs, not pmfs
- Markov property useful for same reasons as in discrete time/state
  - ⇒ But not that useful as in discrete time /state
- More details later

Gaussian processes

- $X(t)$ is a Gaussian process when all prob. distributions are Gaussian
- For arbitrary $n > 0$, times $t_1, t_2, \ldots, t_n$ it holds
  - Values $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly Gaussian RVs
- Simplifies study because Gaussian distribution is simplest possible
  - Suffices to know mean, variances and (cross-)covariances
  - Linear transformation of independent Gaussians is Gaussian
  - Linear transformation of jointly Gaussians is Gaussian
- More details later

Mean and variance of biased steps

- Define step RV $Y_n =\quad 1$, with probabilities

$$P(Y_n = 1) = \frac{1}{2}\quad 1 + \frac{\ }{\ }\sqrt{h}\quad ;\quad P(Y_n = \quad 1) = \frac{1}{2}\quad 1$$

Central Limit Theorem as $h \to 0$

- Consider time $T = Nh$, index $n = 1, 2, \ldots, N$. Write $X(nh)$ as

$$X(nh) = X((n-1)h) + \sqrt{h}\;Y_n$$

- Use recursively to write $X(T) = X(Nh)$ as

$$X(T) = X(Nh) = X(0) + \sqrt{h}\sum_{n=1}^{N} Y_n = \sqrt{h}\sum_{n=1}^{N} Y_n$$

- As $h \to 0$ we have $N \to \infty$ and $\sum_{n=1}^{N} Y_n$ normally distributed
- As $h \to 0$, $X(T)$ tends to be normally distributed by CLT
  - Need to determine mean and variance (and only mean and variance)

Mean and variance of $X(T)$

- Expected value of $X(T)$ = scaled sum of $E[Y_n]$ (recall $T = Nh$)

$$E[X(T)] = \sqrt{h} \quad N \quad E[Y_n] = \sqrt{h} \quad N \quad \frac{}{} \sqrt{h} = \quad T$$

- Variance of $X(T)$ = scaled sum of variances of independent Y_n

$$\operatorname{var}[X(T)] = \sqrt{h}^{\,2} \quad N \quad \operatorname{var}[Y_n]$$

$$= {}^2h \quad N \quad 1 \quad \frac{2}{2}h \quad ! \quad {}^2T$$

  - Used $T = Nh$ and $1 \quad ({}^2 = {}^2)h \; ! \quad 1$
- Brownian motion with drift (BMD)) $X(t) \quad N \quad t; \; {}^2t$
  - Normal with mean t and variance ${}^2t$
  - Independent and stationary increments

Geometric random walk

- Suppose next state follows by multiplying current by a random factor
  - Compare with adding or subtracting a random quantity
- Define RV $Y_n = \quad 1$ with probabilities as in biased random walk

$$P(Y_n = 1) = \frac{1}{2}\left(1 + \frac{\;}{\;}\sqrt{h}\right); \quad P(Y_n = \quad 1) = \frac{1}{2}\left(1 \quad \frac{\;}{\;}\sqrt{h}\right)$$

- Def: The geometric random walk follows the recursion

$$Z(nh) = Z((n \quad 1)h)e^{\quad \sqrt{h}\, Y_n}$$

  - When $Y_n = 1$ increase $Z(nh)$ by relative amount $e^{\quad \sqrt{h}}$
  - When $Y_n = \quad 1$ decrease $Z(nh)$ by relative amount $e^{\quad \sqrt{h}}$
- Notice $e^{\quad \sqrt{h}} \quad 1 \quad \sqrt{h}$) Useful to model investment return

Geometric Brownian motion

- Take logarithms on both sides of recursive definition

$$\log\big(Z(nh)\big) = \log\big(Z((n-1)h)\big) + \sigma\sqrt{h}\,Y_n$$

- Define $X(nh) = \log\big(Z(nh)\big)$, thus recursion for $X(nh)$ is

$$X(nh) = X((n-1)h) + \sigma\sqrt{h}\,Y_n$$

  ⇒ As $h \to 0$, $X(t)$ becomes BMD with parameters μ and σ^2

- Def: Given a BMD $X(t)$ with parameters μ and σ^2, the process $Z(t)$

$$Z(t) = e^{X(t)}$$

  is a geometric Brownian motion (GBM) with parameters μ and σ^2

White Gaussian noise

Dirac delta function

- Consider a function $_h(t)$ defined as

$$_h(t) = \begin{cases} 1/h & \text{if} \quad h/2 \quad t \quad h/2 \\ 0 & \text{else} \end{cases}$$

- “Define” delta function as limit of $_h(t)$ as $h \to 0$

$$(t) = \lim$$

Heaviside's step function and delta function

- Integral of delta function between ∞ and t

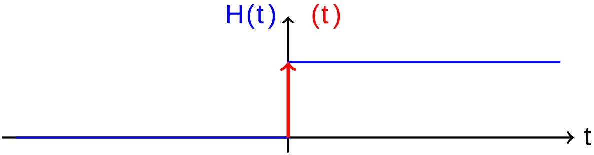
$$\int_{\infty}^{t} (u)\,du = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t > 0 \end{cases} := H(t)$$

  ⇒ H(t) is called Heaviside's step function

- Define the derivative of Heaviside's step function as

$$\frac{\partial H(t)}{\partial t} = (t)$$

  ⇒ Maintains consistency of fundamental theorem of calculus



Properties of white Gaussian noise

- For different times t_1 and t_2, $W(t_1)$ and $W(t_2)$ are uncorrelated

$$E[W(t_1)W(t_2)] = R_W(t_1, t_2) = 0, \quad t_1 \neq t_2$$

- But since $W(t)$ is Gaussian uncorrelatedness implies independence
  - ⇒ Values of $W(t)$ at different times are independent
- WGN has infinite power ⇒ $E\left[W^2(t)\right] = R_W(t, t) = \sigma^2 \delta(0) = \infty$
  - ⇒ WGN does not represent any physical phenomena
- However WGN is a convenient abstraction
  - Approximates processes with large power and independent samples
- Some processes can be modeled as post-processing of WGN
  - ⇒ Cannot observe WGN directly
  - ⇒ But can model its effect on systems, e.g., filters

Mean-square derivative of a random process

- Consider a realization $x(t)$ of the random process $X(t)$
- Def: The derivative of (lowercase) $x(t)$ is

$$\frac{\partial x(t)}{\partial t} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

- When this limit exists $\Rightarrow$ Limit may not exist for all realizations
- Can define sure limit, a.s. limit, in probability, ...
  - $\Rightarrow$ Notion of convergence used here is in mean-squared sense
- Def: Process $\partial X(t)/\partial t$ is the mean-square sense derivative of $X(t)$ if

$$\lim_{h \to 0} E\left[\left(\frac{X(t+h) - X(t)}{h} - \frac{\partial X(t)}{\partial t}\right)^2\right] = 0$$

Mean-square integral of a random process

- Likewise consider the integral of a realization $x(t)$ of $X(t)$

$$\int_a^b x(t)dt = \lim_{h \to 0} \sum_{n=1}^{(b-a)/h} hx(a+nh)$$

  - ⇒ Limit need not exist for all realizations

- Can define in sure sense, almost sure sense, in probability sense, . . .
  - ⇒ Again, adopt definition in mean-square sense

- Def: Process $\int_a^b X(t)dt$ is the mean square sense integral of $X(t)$ if

$$\lim_{h \to 0} E \left[\sum_{n=1}^{(b-a)/h} hX(\right.$$

Linear state model example

- Def: A random process $X(t)$ follows a linear state model if

$$\frac{@X(t)}{@t} = aX(t) + W(t)$$

with $W(t)$ WGN, autocorrelation $R_W(t_1; t_2) = {}^2 (t_1 \quad t_2)$

- Discrete-time representation of $X(t)$) $X(nh)$ with step size h
- Solving differential equation between nh and $(n+1)h$ (h small)

$$X((n+1)h) \quad X(nh)e^{ah} + \int_{nh}^{(n+1)h} W(t)\,dt$$

- Defining@581 179.872 1md4 g 0 G /F38 8.96638 Tf 1@m cm- 8.96638 Tf 1 0 0

Glossary

- Markov process
- Gaussian process
- Stationary process
- Gaussian random vectors
- Mean vector
- Covariance matrix
- Multivariate Gaussian pdf
- Linear functional
- Autocorrelation function
- Brownian motion (Wiener process)
- Brownian motion with drift
- Geometric random walk
- Geometric Brownian motion
- Investment returns
- Dirac delta function
- Heaviside's step function
- White Gaussian noise
- Mean-square derivatives
- Mean-square integrals
- Linear (vector) state model